12.3.2 Replacement Section

If \( x \) is a variable, then an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots
\]

is called a power series (centered at \( x = 0 \)).

\[
\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots + a_n (x-c)^n + \cdots
\]

is a power series centered at \( x = c \), where \( c \) is a constant.

The equal sign above means that the left side equals the right side for all values in the domain. This means the above is an identity. But for \textit{what} values of \( x \) does the identity hold? We have to find them. For all such \( x \) values, we say the series converges. For the values of \( x \) for which the identity is NOT true, we say the series diverges.

For a power series centered at \( x = c \), exactly one of the following is true:

1) The series converges only at \( x = c \). (All power series converge at their center!!)
2) The series converges for all \( x \).
3) There exists an \( R > 0 \) such that the series converges for \( |x - c| < R \) and diverges for \( |x - c| > R \).

\( R \) is called the radius of convergence of the power series.

In part 1) the radius is 0.
In part 2), the radius is \( \infty \).
In part 3) The corresponding domain, \( (c-R, c+R) \), is called the interval of convergence or the domain of the power series.

Note: to determine if the endpoints are included or not, we must test each endpoint independently.

Note 2: We typically use the \textit{ratio test} to determine the radius of convergence.

Let's do one together:

Example 1:

Based on the fact that a 4\textsuperscript{th} degree Maclaurin polynomial for \( f(x) = e^x \) is \( M_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \),

find the \( n \)th term, then find the radius and interval of convergence for the representative power series.

\[
\text{\( n \)th term : } \frac{x^n}{n!}
\]

\text{Radius} = \infty

\text{Interval of convergence} : (-\infty, \infty)
Now, work with your team on the following problems:

Often, we will be dealing with power series representing unknown functions. While we may not recognize the function the series actually represents, we can still determine the values of x for which it does represent the unknown function.

Example 2:
Find the radius of convergence and the interval of convergence. Be sure to test the endpoints independently.

(a) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 5)^n \)

Converges when \( |x-5| < 2 \)

\( R = 2 \)
Interval: \( (3, 7] \)

(Don’t forget to check endpoints!)

(b) \( \sum_{n=0}^{\infty} \frac{x^n}{3^n} \)

Converge when \( |r| = \left| \frac{x}{3} \right| < 1 \)
\( |x| < 3 \)
\( |x-0| < 3 \)

\( R = 3 \)
Interval: \(-3 < x < 3\)

No need to test endpoints since it is geometric.

Watch out: The \( n+1 \) term here is not \( 2n+1 \)

(c) \( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \)

Converges for all \( x \)

\( R = \infty \)
Interval: \( (-\infty, \infty) \)

(d) \( \sum_{n=0}^{\infty} n(x-3)^n \)

Diverges by \( n^{th} \) term divergence test

\( R = 0 \)

"Interval" is only \( x = 3 \)

Hint: You might want to look closely and THINK about this one before jumping straight to the Ratio Test.
Fun (mostly review) Facts:

Taylor Series centered at \( x = c \):

\[
f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots
\]

Once again, if \( c = 0 \), the series is called a Maclaurin series.

Notice we now use an equal sign instead of an approximation sign. Do you know why???

There are four special Maclaurin series you must know. These are the series for \( e^x \), \( \sin x \), \( \cos x \), and \( \frac{1}{1-x} \). These series, under the operations of calculus, behave like the functions they represent on their interval of convergence. For series with a finite interval of convergence, such as the series for \( \frac{1}{1-x} \), taking the derivative or integral will not change the radius of convergence but may change the endpoints of the interval of convergence.

Once we have these series memorized, we can conveniently manipulate them to suit other similar non-polynomial functions as we did in the previous section with Taylor polynomials.

You can manipulate these three special series (or any series we are given) to find other series by using the following techniques. Note: the radius of convergence may change, though:

1) Substitute into a series for \( x \)
2) Multiply or divide the series by a constant and/or a variable
3) Add or subtract two series
4) Differentiate or integrate a series (may change the interval, but not the radius of convergence)
5) Recognize the series as the sum of a geometric power series (next section)

Try these:

Example 4:
If \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \), find \( f'(x) \) and \( f'(0) \). Do you recognize this familiar function?

\[
f'(x) = 0 + 1 \cdot 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots
\]

\[
f'(0) = 1 + 0 + 0 + \cdots = 1 \quad \text{(<- makes sense),} \quad e^0 = 1\]

Example 5:
If \( f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \), and \( F(x) = \int f(x)\,dx \), find \( F(x) \). Do you recognize this familiar function?

Here's a little hint: You are integrating. Don't forget the +C!!!

\[
F(x) = \left[ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right] + \text{c}
\]

given \( F(0) = 1 \), so \( F(x) = 1 - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \cdots + \frac{(-1)^{n+1}x^{2n+2}}{(2n+1)!} + \text{c} \)

\[
\text{Hmm... alternating... plus... looks almost like \( \sin x \) but the signs are wrong...}
\]

\[
F(x) = -\sin x
\]
### Table of Useful Maclaurin Series

<table>
<thead>
<tr>
<th>Series</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1-z} )</td>
<td>( \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots )</td>
</tr>
<tr>
<td>( \frac{1}{1+z} )</td>
<td>( \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \ldots )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots )</td>
</tr>
<tr>
<td>( \ln(1+x) )</td>
<td>( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots )</td>
</tr>
<tr>
<td>( \arctan x )</td>
<td>( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots )</td>
</tr>
</tbody>
</table>

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**Now let’s look specifically at Geometric Series:**

A particularly important skill to develop for the AP exam, other than checking that you’re in Radian mode, is to represent certain types of rational functions as a geometric series. Rather than producing the power series using Taylor’s Rule, you will want to develop the series by manipulating a geometric series or, in some cases, using Long Division.

**Example 1:**
First we’ll do a quick review of geometric series. Geometric series are formed by multiplying by a common ratio \( r \).

(a) Suppose I told you to start with \( a_1 = 1 \) and so let \( r = \frac{2}{3} \), what geometric series would you write? What would the sum be?

\[
S = \frac{1}{1-\frac{2}{3}} = \frac{3}{1} = 3
\]

(b) What if \( a_1 = 1 \) and \( r = -\frac{2}{3} \)?

\[
S = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}
\]

(c) What if \( a_1 = 1 \) and \( r = x \)?

\[
S = \frac{1}{1-x}
\]

For \( |x| < 1 \), it looks familiar? Yes! This is \( \frac{1}{1-x} \), or the Big Y!
Example 2:
Verify your answer from Example 1(c) by finding the power series for \( \frac{1}{1-x} \) centered at \( c = 0 \) by
(a) using Taylor's Rule
(b) using L'Hôpital's Rule
(c) Find the radius and interval of convergence. Verify by graphing.

\[ \frac{1}{1-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + x^n + \ldots \]

\( f(x) = \frac{1}{1-x} \quad \Rightarrow \quad f(0) = 1 \)
\( f'(x) = -1(1-x)^{-2} \cdot -1 \quad \Rightarrow \quad f'(0) = 1 \)
\( f''(x) = 2(1-x)^{-3} \cdot -1 \quad \Rightarrow \quad f''(0) = 2 \)
\( f'''(x) = -6(1-x)^{-4} \cdot -1 \quad \Rightarrow \quad f'''(0) = 6 \)

\( \text{Same series!} \)

\( \text{Geometric} \quad \Rightarrow \quad |r| = |x| < 1 \)
\( \Rightarrow \quad |x-0| < 1 \)
\( \Rightarrow \quad |x| < 1 \)
\( \Rightarrow \quad 2 = 1 \)
\( \text{Interval:} \quad (-1, 1) \)

For Examples 3-6: When you are done, check your answer using Desmos. First graph the given function and then type in the power series and watch what happens as you enter more and more terms. You can actually see why it is called "the interval of convergence!"

Example 3:
Find a power series for \( \frac{1}{1+x} \) centered at \( c = 0 \), then find the interval of convergence. Include the first four nonzero terms and the general term.

Use MAC for \( \frac{1}{1+x} \), substitute \( -x \) in for \( x \)

\[ \frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots + (-x)^n + \ldots \]
\[ = 1 - x + x^2 - x^3 + \ldots \]

Example 4:
Find a power series that represents \( \frac{x}{1+x} \) centered at \( c = 0 \), then find the interval of convergence. Include the first four nonzero terms and the general term.

We can use the above and multiply by \( x \)

\[ \frac{x}{1+x} = x \left( 1 - x + x^2 - x^3 + \ldots \right) \]
\[ = x - x^2 + x^3 - x^4 + \ldots \]

Ratio Test:
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x| \]
\[ |x| < 1 \]
\[ \Rightarrow \quad \text{Interval:} \quad (-1, 1) \]

Example 5:
Find a power series for \( f(x) = \frac{1}{1-x^2} \) centered at \( c = 0 \), then find the interval of convergence. Find the first four nonzero terms and the general term.

Use MAC for \( \frac{1}{1+x} \) and substitute \( x^2 \) for \( x \)

\[ \frac{1}{1-x^2} = 1 + (x^2)^2 + (x^2)^3 + \ldots + (x^2)^n + \ldots \]
\[ = 1 + x^2 + x^4 + x^6 + \ldots + x^{2n} + \ldots \]

\( \text{Rearranged:} \quad \text{Min. } r = x^2 \)
\[ 1 + x + x^2 + x^3 + \cdots + x + \cdots \]

Geometric series with \( r = x^2 \).
\[ \text{Converges when } |r| = |x^2| < 1 \]
\[ -1 < x < 1 \]
\[ \text{Interval: } (-1, 1) \]

**Note:** Geometric series do not need to test endpoints.

One last one to try . . . But, watch out! . . .

When you replace \( x \) with a multiple of \( x \), beware a change in the radius and interval of convergence . . .

**Example 6:**
Find a power series that represents \( \frac{1}{1-2x} \) centered at \( c = 0 \), then find the interval of convergence. Include the first four nonzero terms and the general term.

Use Maclaurin for \( \frac{1}{1-x} \) and sub \( 2x \) in for \( x \)

\[ \frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots + (2x)^n + \cdots \]
\[ = 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^n x^n + \cdots \]

Geometric with \( r = 2x \)
\[ \text{Converges when } |r| = |2x| < 1 \]
\[ |x| < \frac{1}{2} \]
\[ \text{Radius: } R = \frac{1}{2} \]
\[ \text{Interval: } -\frac{1}{2} < x < \frac{1}{2} \]
(no need to test endpoints)
Things get a little more complicated (and definitely more time-consuming) when finding Intervals of Convergence for TAYLOR Series not centered at zero. Can you use any of those tricks (substitution, differentiation, integration) to make things go faster? NOPE.

Give these problems a try. Do the problems in the FIRST COLUMN now. The second column will be assigned later.

For the following problems: a) find the Taylor series expansion for \( f \) about \( x = a \); b) express the series in Sigma notation; and c) find the interval of convergence.

1546. \( f(x) = \frac{1}{x} \quad a = 3 \)
1547. \( f(x) = \cos x \quad a = \frac{\pi}{2} \)
1548. \( f(x) = \frac{1}{x+2} \quad a = 3 \)
1549. \( f(x) = \ln x \quad a = 1 \)

1550. \( f(x) = \sin(x) \quad a = \frac{\pi}{6} \)
1551. \( f(x) = \frac{1}{x^2} \quad a = 1 \)
1552. \( f(x) = e^x \quad a = 2 \)
1553. \( f(x) = 2^x \quad a = 1 \)

1546. \( \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{3}{2} \right) (x-3)^{n-1} \quad \text{Interval of Conv: } \quad 0 < x < 6 \)
1547. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x - \frac{\pi}{2})^{n-1}}{(2n-1)!} \quad \text{Converges for } \quad (0, \pi) \)
1548. \( \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{5 n+1} \quad \text{Interval of Conv: } \quad (-2, 8) \)
1549. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^{n-1}}{n} \quad \text{Interval of Conv: } \quad (0, 2] \)

Now you get to enjoy all of these tools!

1555. Find the Maclaurin series for \( F(x) = \sqrt{1 + x} \).

\[ \sqrt{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \ldots \]

1556. Let \( f \) be a function that has derivatives of all orders for all real numbers. Assume \( f(0) = 0, f'(0) = 5, f''(0) = -4 \) and \( f^{(n)}(0) = 30 \). Write the third order Taylor polynomial for \( f \) centered at \( x = 0 \) and use it to approximate \( f(0.3) \).

\[ f(x) \approx 9 + 5x - \frac{5}{2} \frac{x^2}{2} + \frac{5}{3} \frac{x^3}{3} \]

\[ f(0.3) \approx 10.482 \]

1559. Find the Taylor polynomial of order 3 generated by \( f(x) = \cos 3x \) centered at \( x = \frac{\pi}{6} \).

Now, the problem.

\[ \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n n! (1 - 2n)} x^n \]

Answers:

1555. 1 + \frac{1}{2} + \frac{5}{16} + \frac{5}{32} + \ldots

1556. 9 + 5x - 2x^2 + 6x^3

1559. -1 + (x - \frac{\pi}{6})^2

Note: This answer is sufficient. Finding the general term is tough. But for the record here it is in Sigma form:

\[ (x + 1)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n n! (1 - 2n)} x^n \]
\[ f(\frac{\pi}{6}) \approx 10.982 \]

1559. Find the Taylor polynomial of order 3 generated by \( f(x) = \cos 3x \) centered at \( x = \frac{\pi}{6} \).

\[
\cos(3x) \approx -1 + \frac{9(x - \frac{\pi}{6})^2}{2}
\]

\[
\cos(3x) \approx -1 + \frac{1}{3} (3x - \pi)^2
\]

Still want more practice? Try these optional problems:

1542. Using variable substitution, identities, differentiation, or integration on the series in the table above, find series representations for each of the following functions.

a) \( \frac{1}{1 + x} \)  
   c) \( xe^x \)  
   e) \( \sin^2 x \)  
   g) \( e^x \)

b) \( \sin 2x \)  
   d) \( \cos^2 x \)  
   f) \( \frac{x^2}{1 - 2x} \)  
   h) \( \int e^x \, dx \)
\[ \frac{1}{1 + x^2} = 1 - (x^2) + (x^4) - (x^6) + \cdots \]

\[ = 1 - x^2 + x^4 - x^6 + \cdots \]

1) \[ \sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \]

\[ 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \]

c) \[ x e^x = x \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) \]

\[ = x + x^2 + x^3 + x^4 + \frac{x^5}{4!} + \cdots \]

\[ = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \]

d) \[ \cos^2 x \rightarrow 2\cos^2 x - 1 = \cos(2x) \]

\[ \cos^2 x = \frac{1}{2}(\cos(2x) + 1) \]

\[ \cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \]

\[ \frac{1}{2}(\cos(2x) + 1) = \frac{1}{2} \left[ 1 + \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right) \right] \]

\[ = \frac{1}{2} \left[ 2 - (2x)^2 + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right] \]

\[ = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1)^n (2x)^{2n}}{(2n)!} \]

e) \[ \sin^2 x \rightarrow 1 - 2\sin^2 x = \cos(2x) \]

\[ \sin^2 x = \frac{1}{2} \left( 1 - \cos(2x) \right) \]

\[ \frac{1}{2} \left( 1 - \cos(2x) \right) = \frac{1}{2} \left[ 1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right) \right] \]

\[ = \frac{1}{2} \left[ \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots \right] \]

\[ = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{(2n)!} \]

\[ \frac{x^n}{1 - 2x} = x \left( \frac{1}{1 - 2x} \right) = x \left[ 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + \cdots \right] \]

\[ = x^2 + 2x^3 + 4x^4 + 8x^5 + 16x^6 + \cdots \]

\[ = \sum_{n=1}^{\infty} (2)^{n-1} x^n \]

Is this OK?
\[ e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \int e^x \, dx = \int \left( 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right) \, dx \]

\[ = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \]

Do we need to add + C?